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ON SOME CONSTRUCTIONS OF OPTIMUM CHEMICAL BALANCE WEIGHING DESIGNS

BY

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Abstract. Theorem giving the necessary and sufficient conditions under which a chemical balance weighing design for $v+1$ objects is optimal is proved. Also certain new construction methods of these optimum designs by utilizing the incidence matrices of BIB designs for v treatments are given.

1. Introduction. Let us consider the problem of weighing p objects in n weighings on a chemical balance. The corresponding design matrix X has entries -1, 1, or 0 if, respectively, the object is kept in the left or right pan, or is not included in the particular weighing. The least squares estimator of the vector of the true weights is given by

$$\hat{w} = (X'X)^{-1}X'y,$$

and the covariance matrix of $\hat{w}$ is $\sigma^2(X'X)^{-1}$, provided $X'X$ is nonsigular. Here w and y are the column vectors of the unknown weights of the p objects and of the observations in the n weighings, respectively.

Definition 1.1 (Hotelling [2]). A weighing design is said to be *optimal* if $\operatorname{Var}(\hat{w}_i) = \sigma^2/n$ for $i = 1, \ldots, p$, i.e. if $X'X = nI_p$, I_p being the $p \times p$ identity matrix.

The problem is to choose X in such a way that the variance factors are minimized. Several methods of constructing X are available in the literature. Dey [1], Saha [5], Kageyama and Saha [3] and others have shown how optimum chemical balance weighing designs can be constructed from the incidence matrices of balanced incomplete block (BIB) designs for $p = v$ objects. Saha and Kageyama [6] have constructed optimum chemical balance weighing designs for $v+1$ objects in $4(r-\lambda)$ weighings from incidence matrices of BIB designs for v treatments.

In the present paper we study some other methods of the construction of the design matrix X for a chemical balance weighing design problem ($p = v+1$ objects) using the incidence matrices of some BIB designs for v treatments, which give new optimum chemical balance weighing designs.

2. Optimum weighing designs for $v+1$ objects. Let N_i^* be the incidence matrix of BIB design with parameters v, b_i, r_i, k_i, λ_i, $i = 1, 2$, and $r_1 > \lambda_1$, $r_2 \geqslant \lambda_2$. From N_i^* we obtain another matrices $N_i = 2N_i^* - \mathbf{1}_v \mathbf{1}'_{b_i}$, $i = 1, 2$, where $\mathbf{1}_a$ is the $a \times 1$ vector of 1's. Now, we define the matrix X as

$$X_j = \begin{bmatrix} N'_1 & \mathbf{1}_{b_1} \\ N'_2 & (-1)^j \mathbf{1}_{b_2} \end{bmatrix}, \quad j = 1, 2. \tag{2.1}$$

In these desings we have $p = v+1$ and $n = b_1 + b_2$.

Theorem 2.1. *The chemical balance weighing desings with X_j given by* (2.1) *are optimal if and only if*

$$b_1 + b_2 = 4(r_1 - \lambda_1) + 4(r_2 - \lambda_2) \tag{2.2}$$

and

$$b_1 + (-1)^j b_2 = 2[r_1 + (-1)^j r_2], \quad j = 1, 2. \tag{2.3}$$

Proof. If X_j is given by (2.1), then

$$X_j X'_j = \begin{bmatrix} (b_1 + b_2 - \lambda) I_v + \lambda \mathbf{1}_v \mathbf{1}'_v & a_j \mathbf{1}_v \\ a_j \mathbf{1}'_v & b_1 + b_2 \end{bmatrix},$$

where $\lambda = b_1 + b_2 - 4(r_1 - \lambda_1) - 4(r_2 - \lambda_2)$, while $a_j = -[b_1 + (-1)^j b_2] + 2[r_1 + (-1)^j r_2]$, $j = 1, 2$. Thus $X'X = nI_p$ exactly when (2.2) and (2.3) hold.

Conditions (2.2) and (2.3) yield the following corollaries.

Corollary 2.1. *If a chemical balance weighing design with X_1 given by* (2.1) *is optimal, then $k_1 \neq k_2$ and $v \neq k_1 + k_2$ or $k_1 = k_2$ and $b_1 = b_2$.*

Corollary 2.2. *If a chemical balance weighing design with X_2 given by* (2.1) *is optimal, then $k_1 \neq k_2$ and $v+1 = 2(r_1 k_1 + r_2 k_2)/(r_1 + r_2)$.*

Condition (2.3) is the identity when

(i) $k_1 = k_2$ and $b_1 = b_2$ for $j = 1$;

(ii) N_1^* is the incidence matrix of a BIB design and N_2^* is the incidence matrix of its complementary design for $j = 2$,

and, moreover, for $j = 1, 2$, condition (2.2) is reduced to

$$b_1 = 4(r_1 - \lambda_1). \tag{2.4}$$

The BIB designs for which condition (2.4) holds belong to the family (A) (cf. Definition 5.3.1 of Raghavarao [4]).

Corollary 2.3. *If N_1^* is the incidence matrix of a BIB design belonging to the family (A), then*

(2.5) $$X_j = \begin{bmatrix} N_1' & \mathbf{1}_{b_1} \\ (-1)^{j+1} N_1' & (-1)^j \mathbf{1}_{b_2} \end{bmatrix}, \quad j = 1, 2.$$

are optimum chemical balance weighing designs.

3. A few special cases. First, consider the case $r_2 = \lambda_2$. Then (2.2) and (2.3) are of the form

(3.1) $$b_1 + b_2 = 4(r_1 - \lambda_1)$$

and

(3.2) $$b_2 = (-1)^j b_1 + 2(-1)^{j+1} r_1, \quad j = 1, 2,$$

respectively. Now (3.1) and (3.2) imply $v = 2k_1 \pm 1$, and we arrive at the following result due to Saha and Kageyama [6]:

COROLLARY 3.1. *If there exists a BIB design with the parameters* $v, b_1, r_1, k_1, \lambda_1$, *where* $v = 2k_1 \pm 1$, *then the chemical balance weighing designs with*

(3.3) $$X_j = \begin{bmatrix} N_1' & \mathbf{1}_{b_1} \\ \mathbf{1}_{b_2}\mathbf{1}_v' & (-1)^j \mathbf{1}_{b_2} \end{bmatrix}, \quad j = 1, 2,$$

where b_2 *is given by* (3.2), *are optimal.*

Now suppose that a BIB design with parameters $v, b_1, r_1, k_1, \lambda_1$ is given. The parameters of its complementary design are v, $b_1^* = b_1$, $r_1^* = b_1 - r_1$, $k_1^* = v - k_1$, $\lambda_1 = b_1 - 2r_1 + \lambda_1$ and for these two designs we have obviously the following

THEOREM 3.1. *The parameters* $v, b_1, r_1, k_1, \lambda_1$ *of the BIB design satisfy condition* (3.2) *for* $j = 1$ *if and only if the parameters of its complementary design satisfy condition* (3.2) *for* $j = 2$.

We give now series of BIB designs for which the chemical balance weighing design, with X_1 given by (3.3), is optimal:

(i) The parameters $v = b_1 = 4t+3$, $r_1 = k_1 = 2(t+1)$, $\lambda_1 = t+1$, where $4t+3$ is a prime or a prime power (the complementary design to the design which is described in Theorem 5.7.4, Raghavarao [4]).

(ii) The parameters $v = 4t+1$, $b_1 = 2(4t+1)$, $r_1 = 2(2t+1)$, $k_1 = 2t+1$, $\lambda_1 = 2t+1$, where $4t+1$ is a prime or a prime power (the complementary design to the design which is described in Theorem 5.7.5, Raghavarao, [4]).

(iii) The parameters $v = b_1 = 4k^2 - 1$, $r_1 = k_1 = 2k^2$, $\lambda_1 = k^2$ (provided there exists a BIB design with $\lambda = 1$ and $r = 2k+1$, Raghavarao, [4], Theorem 5.9.2).

(iv) The parameters $v = 2k_1 - 1$, $b_1 = (2k_1-1)!/(k_1!(k_1-1)!)$, $r_1 = (2k_1-2)!/((k_1-1)!)^2$, k_1, $\lambda_1 = (2k_1-3)!/((k_1-1)!(k_1-2)!)$ of an irreducible BIB design (see Raghavarao, [4], p. 90) for all $k_1 \geqslant 2$.

From Theorem 3.1 it is obvious that the parameters of the complementary designs to series (i)–(iv) give an optimum chemical balance weighing design with X_2 given by (3.3).

It can be seen that $b_2 = 1$ in (i) and (iii), $b_2 = 2$ in (ii) and $b_2 = (2k_1-2)!/(k_1!(k_1-1)!)$ in (iv). If $b_2 \geqslant 2$, we have

COROLLARY 3.2. *If there exists a BIB design with parameters $v, b_1, r_1, k_1, \lambda_1$ for which $v = 2k_1+(-1)^j, (-1)^j b_1+2(-1)^{j+1} r_1 \geqslant 2$ ($j = 1, 2$), then a chemical balance weighing design with*

$$X_j = \begin{bmatrix} N_1' & \mathbf{1}_{b_1} \\ \boldsymbol{a}_{b_2}\mathbf{1}_v' & (-1)^j \mathbf{1}_{b_2} \end{bmatrix}, \quad j = 1, 2,$$

is optimal, where b_2 is given by (3.2) *and $\boldsymbol{a}_{b_2}$ is a $b_2 \times 1$ vector with* 1*'s and* -1*'s everywhere.*

Finally, one can easily show that if X is the matrix of an optimum chemical balance weighing design, then DXE is also optimal for every $n \times n$ diagonal matrix $D = \mathrm{diag}\ (\pm 1, \ldots, \pm 1)$ and every $p \times p$ diagonal matrix $E = \mathrm{diag}\ (\pm 1, \ldots, \pm 1)$.

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